**Interacting Field Theories**

Most theories of interest include interactions of some sort. I’ll just take a second to list some possibilities.

**Classical Elastic Field**

First we’ll do the classical elastic field. The free field (homogeneous/isotropic), in 3D is:



Anharmonicities would take the form of interactions between the φ’s. We may generally expect something proportional to λijkφiφjφk + uijkℓφiφjφkφℓ + …. If we still presuppose homogeneity and isotropy, the λ term must be zero, as there is no non-zero rank 3 tensor with such properties. But there is such at fourth order (see Navier Stokes file in Thermo Dynamics). Its general form is:



and the simplest model is I suppose:



And then we may have the following interaction term (implicit summation of course):



But we’ll change the pre-factor 3u → u/4!, ‘cause it makes the PT rules come out nicer. So our simplest interacting model would be:



and we could put an external field in if we want:



and the corresponding Lagrangian density would be:



where the H is to remind us that these are the fully time-developed operators now.

**Interacting Relativistic Bosons**

We can have interacting bosons. Not sure why these bosons are interacting with each other. Maybe we’re attempting some nuclear force model? But whatever. The real ones would have simple interaction terms like this:



The power law φ’s are kind of similar to terms we’ll see in statistical mechanical models which describe coupled spins interacting through the exchange mechanism. We may cut the real boson interaction terms off at the first term perhaps. But while such an odd-powered term may be useful for small deviations from ‘equilibrium’, like how a non-isotropic elastic Lagrangian could include an interaction term (λ/3!)φ(x)3, it cannot by itself be trusted for large deviations because the model would suggest the energy could be lowered indefinitely by making φ as positive as desired. So ultimately, we would need to include the φ4 term, or some such, if we wish to study large deviations. The complex boson would have interactions like as follows:



Note our complex boson interactions would need to be of the form (φ\*φ)m because we’d want to preserve U(1) symmetry by which we have charge conservation.

**Classical EM Interaction**

So recall in the Electrodynamics file we wrote down an action for the EM field [in ‘Gaussian units’]. It was basically,



which was simplified to:



And then we generalized it a bit to incorporate the particles self-consistently, defining:



and adding a KE term for the particles, to write:



And we showed that all the correct equations of motion, for both the particles and the fields, followed from this L. Then we constructed H from this L, and got:



which looks as we expect. Then we put the E field in terms of the space-time potentials, and simplified this to, in the Coulomb gauge:



We can do some more profitable manipulations. We’ll put B in terms of A, and write:



Then we’ll quantize the theory by promoting the classical variables to operators,



And fill in our result for A(x) in ‘Natural Gaussian’ units (since we were using ‘Gaussian’ units when wrote down L, H in the EM file), also using Cond Matt phase convention FWIW:



Now work out the last two terms. And we’ll set t = 0 since H should be independent of time anyway. FYI, that | | means magnitude of vector – basically dot product – and doesn’t imply complex conjugation should be taking place. So |**A**|2 = **A**·**A**, not **A**·**A**\*. But doesn’t actually matter,



where we use ωk = k. And now use,



to get:



Now doing the dot product, and using orthogonality relation between ei**k**·**r**’s we have:



Now using **ε**(-**k**,λ) = -**ε**(**k**,λ), and keeping in mind that **ε**(**k**,λ)·**ε**(**k**,λ´) = δλλ´,



Okay, the signs are not working out. And I’m done trying to figure out why. I get the same problem when I use the normal QFT phase convention for the a’s. Oh well. I actually did this calculation anyway in the photon file appendix, but without ‘simplifying’ A first – maybe that’s where my problem is? And I found (translating a´s → b’s):



And so that brings our H to:



and yeah () is indeed (note the **r** in **A**(**r**,t) is an operator):



This H is often used when we need to treat the EM field quantum mechanically, i.e., when we can have individual photon absorption, etc., but don’t need to worry about particle creation/annihilation. We can use this guy when treating spontaneous decay of atoms from excited states to ground states, for instance, which happens because electrons in excited states will interact with virtual photons bubbling in and out of the EM vacuum.

**Relativistic EM Interaction**

If we do possibly have particle creation/annihilation, then we need to go to the full-blown relativistic action. The interacting photon-electron (or whatever charge) L would be obtained by straightforwardly quantizing the classical EM action, which in Natural Lorentz-Heaviside units, is:



Recalling that the current operator, for a fermion field is:



where e is the electron’s absolute value charge, we can get the full interacting Lagrangian by just adding the fermion Lagrangian and the EM field Lagrangian. So we have (including that Feynman gauge term thing at the end):



I’m a little perplexed though, because the interacting electron-photon Lagrangian seems to be merely quadratic, and would thereby be completely diagonalizable by some linear transformation of variables? No it’s cubic. I presume the classical EM Lagrangian/Hamiltonian would emerge from this guy in the non-relativistic limit.

In any event, I suppose we’ll want to take these models and extract their excitations AFAP. We’ll also consider scattering experiments (note non-interacting particles wouldn’t scatter because they wouldn’t interact so it only makes sense to study scattering with an interacting model), which would be the practical means through which information about the interactions/excitations would be extracted. Given the complexity of the HS, it would appear these things are most conveniently investigated through the machinery of Green’s Functions. So we’ll investigate that next.